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Geometric transformations and new integrable problems of rigid body dynamics

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Abstract. The problem of motion of a rigid body about a fixed point under the action of conservative forces is considered in the case admitting a linear integral but no axis of symmetry—neither in space nor in the body—is present. A simple transformation of the configuration space is used to reduce the problem of motion of the body to another problem concerning the same body under a system of axisymmetric forces. This analogy enables the construction of several new integrable cases of the first problem by transforming certain known ones of the second. The new cases usually involve singular potential terms. Integrals of motion and physical interpretation are given explicitly for one generally integrable case. Other general and conditional cases are pointed out.

1. Introduction

The model of the rigid body and its generalization—the gyrostat—have found a wide range of applications in various fields of physics, in addition to its classical applications in mechanics and astronomy. For example, the gyrostat was used as a model of the earth that takes account of some stationary transport processes on it [1], as a model of the atmosphere and of rotating fluid (e.g. [2]) and as a controlling device in satellite dynamics (e.g. [3]). The rigid body model has also been used in nuclear physics and in optics of nonlinear media (e.g. [4]) and in nonlinear wave propagation (e.g. [5]). Integrable models play an important role in those studies. It is thus of great importance to construct integrable problems as much as possible and to reveal connections between them.

Recently, several integrable cases of motion of the rigid body and the gyrostat under the action of potential and gyroscopic forces have been found, mainly when those forces admit a common axis of symmetry fixed in space [6, 7]. Certain cases of this type were geometrically transformed to other, physically different but mathematically equivalent, cases which admit axial symmetry in the body system but not in space [8]. In this paper we explore an interesting geometric transformation that reduces certain problems admitting space axial symmetry to problems which have neither a space nor body axis of symmetry and thus to construct new integrable problems of the latter. Although the two types of problems are completely different from the physical point of view, the solution of one of them can always be obtained from that of the other.

Consider a rigid body in motion about its fixed point O. Let $OXYZ$ and $Oxyz$ be two Cartesian coordinate systems, fixed in space and in the body, respectively. Let also $\boldsymbol{\omega} = (p, q, r)$ be the angular velocity of the body and $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ be the unit vectors in the directions of the XYZ axes, all being referred to the body system which we take as the system of the principal axes of inertia.

Those variables can be expressed in terms of Euler angles: ψ the angle of precession about the Z axis, θ the angle of nutation (between the z and Z axes) and the angle of proper rotation φ about the z axis. They have the form

$$\begin{aligned}
\boldsymbol{\alpha} &= (\cos\psi\cos\varphi - \cos\theta\sin\psi\sin\varphi, -\cos\psi\sin\varphi - \cos\theta\sin\psi\cos\varphi, \sin\theta\sin\psi)\\
\boldsymbol{\beta} &= (\sin\psi\cos\varphi + \cos\theta\cos\psi\sin\varphi, -\sin\psi\sin\varphi + \cos\theta\cos\psi\cos\varphi, -\sin\theta\cos\psi)\\
\boldsymbol{\gamma} &= (\sin\theta\sin\varphi, \sin\theta\cos\varphi, \cos\theta)\\
\boldsymbol{\omega} &= (\dot\psi\sin\theta\sin\varphi + \dot\theta\cos\varphi, \dot\psi\sin\theta\cos\varphi - \dot\theta\sin\varphi, \dot\psi\cos\theta + \dot\varphi).
\end{aligned}$$

The problem considered here is the general problem of motion of a rigid body about a fixed point under the action of a combination of conservative potential and gyroscopic forces, described by the Lagrangian

$$L = \tfrac{1}{2}\boldsymbol{\omega}\boldsymbol{I}\cdot\boldsymbol{\omega} + \boldsymbol{l}\cdot\boldsymbol{\omega} - V \tag{1}$$

where $\boldsymbol{I} = \mathrm{diag}(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ is the inertia matrix of the body. The potential V and the vector $\boldsymbol{l}$ depend only on the Eulerian angles through the nine direction cosines α_1, $\alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3$.

The Lagrangian (1) describes a conservative system of three degrees of freedom, which admits the Jacobi integral (the Hamiltonian of the system)

$$I_1 \equiv H = \tfrac{1}{2}\boldsymbol{\omega}\boldsymbol{I}\cdot\boldsymbol{\omega} + V = \text{constant}.$$

We recall that for its complete integration in the sense of Liouville we must obtain two additional integrals of motion in involution with the Jacobi integral.

The equations of motion of a rigid body are usually written in the Euler–Poisson variables $\boldsymbol{\omega}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$. For the present problem this form, corresponding to (1), was found in our work [9] to be

$$\begin{aligned}
&\dot{\boldsymbol{\omega}}\boldsymbol{I} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\boldsymbol{I} + \boldsymbol{\mu}) = \boldsymbol{\alpha}\times\frac{\partial V}{\partial\boldsymbol{\alpha}} + \boldsymbol{\beta}\times\frac{\partial V}{\partial\boldsymbol{\beta}} + \boldsymbol{\gamma}\times\frac{\partial V}{\partial\boldsymbol{\gamma}}\\
&\dot{\boldsymbol{\alpha}} + \boldsymbol{\omega}\times\boldsymbol{\alpha} = \boldsymbol{0} \qquad \dot{\boldsymbol{\beta}} + \boldsymbol{\omega}\times\boldsymbol{\beta} = \boldsymbol{0} \qquad \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega}\times\boldsymbol{\gamma} = \boldsymbol{0}
\end{aligned} \tag{2}$$

where $\boldsymbol{I}$ is the inertia tensor of the body at the fixed point and

$$\begin{aligned}
\boldsymbol{\mu} = \boldsymbol{l} &+ \left(\boldsymbol{\alpha}\times\frac{\partial}{\partial\boldsymbol{\alpha}} + \boldsymbol{\beta}\times\frac{\partial}{\partial\boldsymbol{\beta}} + \boldsymbol{\gamma}\times\frac{\partial}{\partial\boldsymbol{\gamma}}\right)\times\boldsymbol{l}\\
&\equiv \frac{\partial}{\partial\boldsymbol{\alpha}}(\boldsymbol{l}\cdot\boldsymbol{\alpha}) + \frac{\partial}{\partial\boldsymbol{\beta}}(\boldsymbol{l}\cdot\boldsymbol{\beta}) + \frac{\partial}{\partial\boldsymbol{\gamma}}(\boldsymbol{l}\cdot\boldsymbol{\gamma}) - \left(\frac{\partial}{\partial\boldsymbol{\alpha}}\cdot\boldsymbol{l}\right)\boldsymbol{\alpha}\\
&\quad - \left(\frac{\partial}{\partial\boldsymbol{\beta}}\cdot\boldsymbol{l}\right)\boldsymbol{\beta} - \left(\frac{\partial}{\partial\boldsymbol{\gamma}}\cdot\boldsymbol{l}\right)\boldsymbol{\gamma} - 2\boldsymbol{l}.
\end{aligned} \tag{3}$$

2. Problem I

Let the rigid body with axial dynamical symmetry $A = B$ be in motion under the action of forces with potential V and gyroscopic forces characterized by the vector $\boldsymbol{l} = (0, 0, \boldsymbol{l}_3)$, both depending only on the variables $\theta, \varphi - \psi$. In terms of the rotation matrix this allows only the combinations of the direction cosines

$$\alpha_1 - \beta_2 \qquad \alpha_2 + \beta_1 \qquad \gamma_3.$$

The Lagrangian of this system is

$$\begin{aligned} L &= \tfrac{1}{2}[A(p^2+q^2)+Cr^2]+l_3 r - V \\ &= \tfrac{1}{2}[A(\dot\theta^2+\sin^2\theta\dot\psi^2)+C(\dot\psi\cos\theta+\dot\varphi)^2]+(\dot\psi\cos\theta+\dot\varphi)l_3 - V. \end{aligned} \tag{4}$$

We will now introduce a change of variables

$$\theta = 2\Theta \qquad \psi = \Psi - \Phi \qquad \varphi = \Psi + \Phi. \tag{5}$$

This will change (4) to

$$\begin{aligned} L = 2\{&A[\dot\Theta^2+\sin^2\Theta\cos^2\Theta(\dot\Psi-\dot\Phi)^2]+C(\sin^2\Theta\dot\Phi+\cos^2\Theta\dot\Psi)^2\} \\ &+2l_3(\sin^2\Theta\dot\Phi+\cos^2\Theta\dot\Psi)-V. \end{aligned} \tag{6}$$

Note that V and l_3 are now functions of Θ and Φ only. The cyclic integral corresponding to the cyclic variable Ψ is

$$4\cos^2\Theta D\dot\Psi - 4(A-C)\sin^2\Theta\cos^2\Theta\dot\Phi + 2l_3\cos^2\Theta = \text{constant} = f \tag{7}$$

where $D = A\sin^2\Theta + C\cos^2\Theta$. Thus we can write

$$\dot\Psi = \frac{f - 2l_3\cos^2\Theta + 4(A-C)\sin^2\Theta\cos^2\Theta\dot\Phi}{4\cos^2\Theta D}. \tag{8}$$

This can be used to ignore the cyclic variable Ψ and construct the Routhian

$$\begin{aligned} R = \frac{1}{4}(L - f\dot\Psi) &= \frac{1}{2}A\left(\dot\Theta^2+\frac{C}{D}\sin^2\Theta\dot\Phi^2\right)+\sin^2\Theta[f(C-A)+2Al_3]\frac{\dot\Phi}{4D} \\ &-\left[\frac{V}{4}+\frac{(f-2\cos^2\Theta l_3)^2}{32\cos^2\Theta D}\right]. \end{aligned} \tag{9}$$

3. Problem II: the case of axially symmetric forces

When all the forces acting on the body have the Z axis (say) as a common axis of symmetry, the two quantities $V', \boldsymbol{l}'$ (or $\boldsymbol{\mu}'$) will be functions of γ only. For such systems the angle of precession ψ is a cyclic variable. Moreover, we will restrict our consideration to the case where the body exhibits axial dynamical symmetry $A = B$ and the vector $\boldsymbol{l}'$ lies along the axis of dynamical symmetry, i.e. $\boldsymbol{l}' = (0, 0, l_3')$. The Lagrangian of the system will have the form

$$L' = \tfrac{1}{2}[A(p^2+q^2)+Cr^2]+l_3' r - V'. \tag{10}$$

It will be more convenient for our purpose to write this Lagrangian explicitly in terms of the Eulerian angles as generalized coordinates:

$$L' = \tfrac{1}{2}[A(\dot\theta^2+\sin^2\theta\dot\psi^2)+C(\dot\psi\cos\theta+\dot\varphi)^2]+l_3'(\dot\psi\cos\theta+\dot\varphi)-V'. \tag{11}$$

Here V', l_3' are now understood as $V'(\theta,\varphi)$ and $l_3'(\theta,\varphi)$. The cyclic integral for (11) can be written as

$$D'\dot\psi + (C\dot\varphi + l_3')\cos\theta = \text{constant} = f' \tag{12}$$

where $D' = A\sin^2\theta + C\cos^2\theta$. That is

$$\dot\psi = \frac{f' - (C\dot\varphi + l_3')\cos\theta}{D'}. \tag{13}$$

Ignoring ψ we construct the Routhian

$$R' = \frac{1}{2}A\left(\dot\theta^2+\frac{C}{D'}\sin^2\theta\dot\varphi^2\right)+\frac{\dot\varphi}{D'}(fC\cos\theta + Al_3'\sin^2\theta)-\left[V'+\frac{(f'-l_3'\cos\theta)^2}{2D'}\right]. \tag{14}$$

4. Equivalence of the two problems

A look at the two Routhians (14) and (9) reveals that their quadratic parts are identical, with the same role played by the pairs of variables θ, φ and Θ, Φ. Moreover, both Routhians have the same structure. They become completely identical if, in (9), we take

$$l_3 = 2l_3' + \frac{f(A-C)}{2A} + \frac{2Cf'\cos\Theta}{A\sin^2\Theta} \qquad V = 4V' + \frac{fl_3'}{A} - \frac{f^2}{8A\cos^2\Theta} + \frac{f'}{A^2\sin^2\Theta}\left[2f'(A+C) + (Cf - 4Al_3')\cos\Theta + \frac{2Cf'}{\sin^2\Theta}\right] \tag{15}$$

where V', l_3' stand for $V'(\Theta, \Phi)$ and $l_3'(\Theta, \Phi)$.

Thus we have established the mathematical equivalence of problems I and II, in the sense that under (15) both problems reduce to one and the same Routhian and hence have the same time solution as concerns the noncyclic coordinates (Θ, Φ) and (θ, φ).

Returning to the original system (4) through the reverse transformation (5) we write (15) in terms of Euler angles:

$$\begin{aligned} l_3 &= 2l_3'\left(\frac{\theta}{2}, \frac{\varphi-\psi}{2}\right) + \frac{f(A-C)}{2A} + \frac{4Cf'\cos\frac{\theta}{2}}{A(1-\cos\theta)} \\ V &= 4V'\left(\frac{\theta}{2}, \frac{\varphi-\psi}{2}\right) + \frac{f}{A}l_3'\left(\frac{\theta}{2}, \frac{\varphi-\psi}{2}\right) - \frac{f^2}{4A(1+\cos\theta)} + \frac{2f'}{A^2(1-\cos\theta)} \\ &\quad \times \left\{2f'(A+C) + \left[Cf - 4Al_3'\left(\frac{\theta}{2}, \frac{\varphi-\psi}{2}\right)\right]\cos\frac{\theta}{2} + \frac{2Cf'}{1-\cos\theta}\right\}. \end{aligned} \tag{16}$$

The most interesting consequence of this equivalence is that, for any pair (V', l_3') for which problem II is integrable on the level f' of the cyclic integral (12), there corresponds a pair (V, l_3) given by (16) which makes problem I integrable. If problem II is integrable for arbitrary initial conditions, i.e. if we deal with a general integrable case of problem II, then (V, l_3) will involve f' as a free parameter. In general, when $f' \neq 0$ the functions V, l_3 have singularity at $\theta = \pi$. Motions beginning or ending on this singularity should be treated in its vicinity on a limiting basis.

As the cyclic constant f of problem I enters in (15), (16), the integrable cases of this problem constructed in this way are, generally speaking, conditional, i.e. valid only on that single level of the cyclic integral. However, as will be seen later, there are some cases when this analogy leads to the construction of general integrable cases for arbitrary initial conditions.

Some assertion has to be made about the determination of the Eulerian angles for the equivalent problems I and II. Let the reduced problem II have a solution $\theta = S(t)$, $\varphi = F(t)$ on some level f' of (12). The remaining angle ψ is obtained by direct integration of (13). On the other hand, the equivalent case of problem I in its reduced form (9) has the solution $\Theta = S(t)$, $\Phi = F(t)$. The angle Ψ is found by integrating (8) as $\Psi = P(t)$ (say). This generates for the original problem I the solution

$$\begin{aligned} \theta &= 2S(t) \\ \varphi &= P(t) + F(t) \\ \psi &= P(t) - F(t). \end{aligned} \tag{17}$$

Note that this solution is valid on the level f of the cyclic integral of that problem and that the value f enters as a parameter in the expressions (16) for V and l_3 of the same problem.

5. New integrable cases

Equations of motion derived from (9) can be used in any analytical or qualitative study of the motion in problem I, as reduction and possibly simplification of the full equations from (4). Our main objective, however, is to construct all possible integrable cases of problem I which are derivable from the known integrable cases of problem II through the above analogy. An up-to-date list of the latter cases can be found in [7]. One must also note that the present analysis applies only to the cases for which $A = B$. We thus have to add this condition to cases valid for arbitrary moments of inertia. We shall consider in detail below only some examples.

5.1. General integrable cases

5.1.1. Case 1. In [10] a conditional integrable case of problem II was introduced, generalizing earlier results [11–13]. This case (number 2 in table 2 of [7]) is characterized by

$$
\begin{aligned}
&A = B = 2C \qquad f' = 0 \\
&l_3' = Ck' \\
&V' = C\left(a_1\gamma_1 + a_2\gamma_2 + b_1(\gamma_1^2 - \gamma_2^2) + 2b_2\gamma_1\gamma_2 + \frac{\lambda}{2\gamma_3^2}\right) \\
&\quad = C\left[a_1 \sin\theta\sin\varphi + a_2\sin\theta\cos\varphi + \frac{1}{2}(1-\cos 2\theta)(b_2\sin 2\varphi - b_1\cos 2\varphi) + \frac{\lambda}{1+\cos 2\theta}\right].
\end{aligned}
\tag{18}
$$

The analogous case of problem I is given by

$$
\begin{aligned}
l_3 &= 2Ck' + \frac{f(A-C)}{2A} \\
V &= 4V'\left(\frac{\theta}{2}, \frac{\varphi-\psi}{2}\right) + \frac{f}{A} - \frac{f^2}{4A(1+\cos\theta)} \\
&= 4C\left\{a_1\sin\frac{\theta}{2}\sin\frac{\varphi-\psi}{2} + a_2\sin\frac{\theta}{2}\cos\frac{\varphi-\psi}{2}\right. \\
&\qquad \left. + \frac{1}{2}(1-\cos\theta)[b_2\sin(\varphi-\psi) - b_1\cos(\varphi-\psi)]\right\} + \frac{4C\lambda - f^2/4A}{1+\cos\theta}
\end{aligned}
\tag{19}
$$

where in the last line we have ignored an nonessential constant.

The formulae (19) characterize an integrable case of problem I. This case is expected to be a conditional one, valid on the level f of the cyclic integral of this problem. In contrast, this is a general integrable case of problem I. In fact, the terms containing the parameter f in (19) can be absorbed through renaming the other arbitrary constants that originate from the original problem I. We can write

$$
\begin{aligned}
l_3 &= k \\
V &= 4C\left\{a_1\sin\frac{\theta}{2}\sin\frac{\varphi-\psi}{2} + a_2\sin\frac{\theta}{2}\cos\frac{\varphi-\psi}{2}\right. \\
&\qquad \left. + \frac{1}{2}(1-\cos\theta)[b_2\sin(\varphi-\psi) - b_1\cos(\varphi-\psi)]\right\} + \frac{CN}{1+\cos\theta}
\end{aligned}
\tag{20}
$$

where k, N are free parameters. This case is generally integrable for arbitrary initial conditions, except for the singularity at $\theta = \pi$. The potential in (20) contains the two terms $a_1\sin\frac{\theta}{2}\sin\frac{\varphi-\psi}{2} + a_2\sin\frac{\theta}{2}\cos\frac{\varphi-\psi}{2}$. Although those terms are single valued for $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \varphi < 2\pi$, $0 \leqslant \psi < 2\pi$, they change sign when any of the angles φ, ψ is increased by 2π, so that the potential becomes double-valued on the natural configuration space of the rigid body. This means that the full integrable model (20) can be used only for problems in which the body does not perform complete rotations around any of the z or Z axes.

In principle, the complementary integral can be obtained from that of (18) (see, e.g., [7]) in an obvious—but very involved—way. We will here construct this integral directly from the Euler–Poisson equations (2), but only for a special version of (20), when $a_1 = a_2 = 0$. It can be easily verified that (2) in which

$$\begin{aligned} \boldsymbol{\mu} &= C(0,0,k) \\ V &= C\left[a(\alpha_1 - \beta_2) + b(\alpha_2 + \beta_1) + \frac{N}{1+\gamma_3}\right] \end{aligned} \tag{21}$$

admit the integral

$$\begin{aligned} I_3 = {} & [p^2 - q^2 - a(\alpha_1 + \beta_2) + b(\alpha_2 - \beta_1)]^2 + [2pq - b(\alpha_1 + \beta_2) - a(\alpha_2 - \beta_1)]^2 \\ & -4k[p(a\alpha_3 + b\beta_3) + q(b\alpha_3 - a\beta_3)] + 2k(r-k)(p^2+q^2) \\ & + \frac{2N}{1+\gamma_3}\left[p^2 + q^2 + kr - k^2 + \frac{N}{2(1+\gamma_3)^2}\right]. \end{aligned} \tag{22}$$

This version involves one parameter N more than a previously known case pointed out in [14]. This parameter invokes the singular potential term $\frac{N}{1+\gamma_3} = \frac{N}{2\cos^2\frac{\theta}{2}}$. A physical interpretation for the integrable pair (21) is possible as follows:

(1) Let an axisymmetric gyroscope with axial moment of inertia I_G placed along the axis of dynamical symmetry of the body be kept rotating with uniform angular velocity Ω. The vector $\boldsymbol{\mu}$ is the gyrostatic moment due to this rotor, provided we take $Ck = I_G\Omega$.

(2) Let the body have total mass M, centre of mass $\boldsymbol{r_0} = (\boldsymbol{x_0}, \boldsymbol{y_0}, \boldsymbol{0})$ lying in its equatorial plane. Suppose the body also contains some magnetized parts with total magnetic moment $\boldsymbol{m} = (\boldsymbol{m_1}, \boldsymbol{m_2}, \boldsymbol{0})$. Let the system be moving in the presence of a uniform gravity field $g\boldsymbol{\alpha}$ and a horizontal magnetic field $H\boldsymbol{\beta}$. Its potential will be

$$g(x_0\alpha_1 + y_0\alpha_2) + H(m_1\beta_1 + m_2\beta_2). \tag{23}$$

This expression can be identified with the first two terms of V if we take $\boldsymbol{m}\cdot\boldsymbol{r}_0 = 0$ and $|g\boldsymbol{r}_0| = |H\boldsymbol{m}|$, so that $Ca = gx_0$, $Cb = gy_0$.

(3) Consider two points $P(0,0,1)$ fixed in the body on the z axis (its axis of dynamical symmetry) and Q on the fixed Z axis at $Z = -1$. The distance $PQ = 2\cos\frac{\theta}{2}$. The singular potential term is thus inversely proportional to the square of the distance PQ.

5.1.2. Case 2. Exactly as in the previous case, we can show that the case

$$\begin{aligned} & A = B = 4C \qquad f = 0 \\ & l_3 = k \\ & V = a\sin\frac{\theta}{2}\sin\frac{\varphi-\psi}{2} + b\sin\frac{\theta}{2}\cos\frac{\varphi-\psi}{2} + \frac{N}{1+\cos\theta} \end{aligned} \tag{24}$$

is a general integrable one. It corresponds to case 1 of table 2 in [7].

5.2. Example of conditional cases

A representative example of the conditional cases is the analogue of the third case of table 2 in [7]. For problem II we have

$$\begin{aligned} & A = B = 2C \qquad f' = 0 \\ & l_3' = 0 \\ & V' = a\sin\theta\sin\varphi + b\sin\theta\cos\varphi + \frac{\varepsilon}{\sin\theta} + \frac{\rho}{2\cos^2\theta}. \end{aligned} \tag{25}$$

This gives for problem I:

$$\begin{aligned} l_3 &= \frac{f}{4} \\ V &= 4\left[a\sin\frac{\theta}{2}\sin\frac{\varphi-\psi}{2} + b\sin\frac{\theta}{2}\cos\frac{\varphi-\psi}{2} + \frac{\varepsilon}{\sin\frac{\theta}{2}} + \frac{\rho - \frac{f^2}{16A}}{1+\cos\theta}\right]. \end{aligned} \tag{26}$$

The parameter f figuring in the potential term can be absorbed in the arbitrary parameter ρ which we rename as σ. We still have f in the expression of l_3, which we rename as k (gyrostatic moment). Thus we can write (26) as

$$\begin{aligned} l_3 &= k \\ V &= 4\left[a\sin\frac{\theta}{2}\sin\frac{\varphi-\psi}{2} + b\sin\frac{\theta}{2}\cos\frac{\varphi-\psi}{2} + \frac{\varepsilon}{\sin\frac{\theta}{2}} + \frac{\sigma}{1+\cos\theta}\right]. \end{aligned} \tag{27}$$

This choice guarantees the integrability of the problem only on the level $f = 4k$ of its cyclic integral.

Similar conditional integrable analogues can be constructed for cases 4 and 6 of table 2 and for cases 4 and 5 of table 1 in [7].

References

[1] Volterra V 1899 *Acta Math.* **22** 201–358
[2] Gluhovsky A B 1999 *Phys. Fluids* **11** 334–3
[3] Hughes P C 1986 *Spacecraft Attitude Dynamics* (New York: Wiley)
[4] Elipe A and Lanchares V 1997 *J. Math. Phys.* **38** 3533–44
[5] Dubrovin B A, Krichever I M and Novikov S P 1986 *Integrable Systems I. Dynamical Systems IV Symplectic Geometry and its Applications* (Berlin: Springer)
[6] Yehia H M 1997 *J. Phys. A: Math. Gen.* **30** 7269–75
[7] Yehia H M 1999 *J. Phys. A: Math. Gen.* **32** 7565–80
[8] Yehia H M 1998 *J. Phys. A: Math. Gen.* **31** 5819–25
[9] Yehia H M 1988 *Celest. Mech.* **41** 275–88
[10] Yehia H M 1996 *Mech. Res. Commun.* **23** 423–7
[11] Yehia H M 1988 *Celest. Mech.* **41** 289–95
[12] Goriatchev D N 1915 *Warshav. Univ. Izvest.* **3** 1–11
[13] Chaplygin S 1948 *Collected Works* vol I (Moscow: Izdat. Tehn. Teor. Lit.) pp 337–46
[14] Yehia H M 1986 *Mech. Res. Commun.* **13** 3 169–72